feature

I dare to find a proof

Area of a Cyclic Quadrilateral

Brahmagupta's Theorem

A surprising but true fact: sometimes a 'low-tech' proof of a theorem is less well-known than the 'hi-tech' one. In this article we see an example of this phenomenon.

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When students ask me a relevant question, I am reminded of this conversation between a mother and child. Child: "Mummy, why are some of your hairs turning grey?" Mother (trying to make best use of the question): "It is because of you, my dear. Every bad action of yours turns one of my hairs grey!" Child (innocently): "Now I know why grandmother has only grey hairs on her head!"

I try to answer relevant questions by students in an appropriate way. When students asked me for the proof of Heron's formula which they had found in their textbook (but without proof), I gave them a proof using concepts they know, similar to the one given in *At Right Angles* (Vol. 1, No. 1, June 2012, page 36), and suggested they look up some internet resources. Later, they told me that they had come across Brahmagupta's formula (for area of a cyclic quadrilateral), had noted its similarity to Heron's formula, but had found the proof used ideas from trigonometry. They asked whether the theorem can be proved using geometry and algebra. I took up the challenge and found such a proof. Here it is.

The theorem is due to the Indian mathematician Brahmagupta (598–670 A.D.) who lived in the central Indian province of Ujjain, serving as the head of the astronomical observatory located there. It was Brahmagupta who wrote the important and influential work *Brahmasphutasiddhānta*. (This is the first mathematical text to explicitly describe the arithmetic of negative numbers and of zero.)

Brahmagupta's formula gives the area of a cyclic quadrilateral (one whose vertices lie on a circle) in terms of its four sides.

Here is the statement of the theorem.

Theorem (Brahmagupta). *If $ABCD$ is a cyclic quadrilateral whose side lengths are a, b, c, d, then its area σ is given by $\sigma = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ where $s = \frac{1}{2}(a+b+c+d)$ is the semi-perimeter of the quadrilateral.*

Note the neat symmetry of the formula. We shall prove it using familiar concepts in plane geometry such as: (i) properties of a circle (ii) properties of similar triangles (iii) Heron's formula for the area of a triangle, according to which the area of a triangle ABC with sides a, b, c is equal to $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$ is half the perimeter of the triangle.

Before offering a proof let us pass the formula through a 'check list' of simple tests.

- *Is the formula dimensionally correct?* Yes; the quantity within the square root is the product of four lengths, so the quantity $\sigma = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ has the unit of area.
- *Is the formula symmetric in the four quantities a, b, c, d?* Yes. (It would be strange if the formula 'preferred' one quantity to another. An example of a formula which is dimensionally correct but not symmetric in a, b, c, d would be the following: $\sqrt{(s-\frac{1}{3}a)(s-\frac{1}{2}b)(s-\frac{1}{4}c)(s-d)}$.)
- *Does the formula give correct results when one side shrinks to zero?* Suppose that $d = 0$. This means that vertices A, D of the quadrilateral have collapsed into each other, and the figure is a triangle (with vertices A, B, C) rather than a quadrilateral. The Brahmagupta formula now reduces to $\sqrt{s(s-a)(s-b)(s-c)}$ which is simply Heron's formula for area of a triangle — a known result.

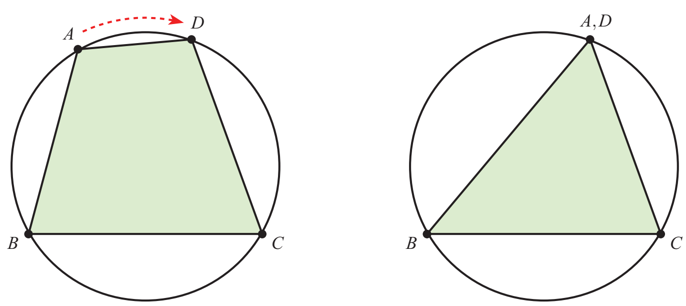

FIGURE 1. Here, vertices A, D have coalesced into each other (hence $d = 0$)

- *Does the formula yield the correct result for a rectangle, which is a special case of a cyclic quadrilateral?* It does: if the rectangle has dimensions $a \times b$, then $s = a + b$, and the formula yields $\sigma = \sqrt{b \cdot a \cdot b \cdot a} = ab$, which is correct.

We see that the formula has passed all the tests; this increases our confidence in it (but of course, these steps are not a substitute for a proof). It is in general a useful exercise to subject a formula to tests like these.

A final comment: the formula gives the area of a *cyclic* quadrilateral in terms of its sides. Implicitly such a formula makes the claim that if the sides of a quadrilateral are fixed, and we are told that the quadrilateral is cyclic, then its area gets fixed. This is so, and it can be proved. For a general quadrilateral there cannot be a formula for area only in terms of its four sides, for the simple reason that the four sides alone cannot fix the quadrilateral. (To see why, think of the quadrilateral as made of four jointed rods having the given lengths. Such a shape is obviously not rigid, so the area is not fixed.)

Proof of the formula

Let $ABCD$ be a cyclic quadrilateral. Since we know that the Brahmagupta formula works for rectangles, there is nothing lost by assuming that $ABCD$ is not a rectangle. In this case at least one pair of opposite sides of the quadrilateral are not parallel to each other. We shall suppose that AD is not parallel to BC, and that lines AD and BC meet when extended at point P as shown in Figure 2. (Under the assumption that AD is not parallel to BC, this will be the case if $AB < CD$. If $AB > CD$, then AD and BC will meet on the 'other' side of the quadrilateral. The third possibility, that $AB = CD$, cannot happen since we have assumed that AD and BC are not parallel to each other.)

Elementary circle geometry shows that $\triangle PAB \sim \triangle PCD$; for we have $\measuredangle PAB = \measuredangle PCD$ and $\measuredangle PBA = \measuredangle PDC$; and the angle at P is shared by the two triangles. Let a, b, c, d be the lengths of AB, BC, CD, DA; let u, v be the lengths of PA, PB; and let e, f be the lengths of the diagonals AC, BD respectively (see Figure 2). Our strategy will now be the following:

Step 1: Find u, v in terms of a, b, c, d, using the similarity $\triangle PAB \sim \triangle PCD$.

Step 2: Find the area of $\triangle PAB$ using Heron's formula.

Step 3: Find the area of $\triangle PCD$, once again using the similarity $\triangle PAB \sim \triangle PCD$.

Step 4: Find the area of the quadrilateral, by subtraction.

Sounds simple, doesn't it? Here's how we execute the steps.

Steps 1 & 2: Let the coefficient of similarity in the similarity $\triangle PAB \sim \triangle PCD$ be k. Since the sides of $\triangle PAB$ are u, v, a, while the corresponding sides of $\triangle PCD$ are $v + b, u + d, c$, we have:

$$v + b = ku, \qquad u + d = kv, \qquad c = ka. \tag{1}$$

Hence we have:

$$k = \frac{c}{a}, \qquad u - v = \frac{b - d}{k + 1} = \frac{a(b - d)}{c + a}, \qquad u + v = \frac{b + d}{k - 1} = \frac{a(b + d)}{c - a}. \tag{2}$$

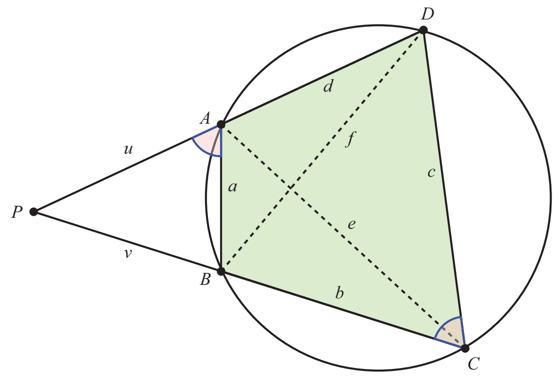

FIGURE 2.

Therefore the semi-perimeter s' of $\triangle PAB$ is given by:

$$2s' = a + u + v = a + \frac{a(b + d)}{c - a} = \frac{a(b + c + d - a)}{c - a}. \tag{3}$$

Hence we have the following relationship between s' and the semi-perimeter s of quadrilateral $ABCD$:

$$s' = \frac{a(s-a)}{c-a}. \tag{4}$$

To compute the area of $\triangle PAB$ we need expressions for the following:

$$2s' - 2a = u + v - a = \frac{a(b+d)}{c-a} - a = \frac{a(d+a+b-c)}{c-a} = \frac{2a(s-c)}{c-a},$$

$$2s' - 2u = a + v - u = a - \frac{a(b-d)}{c+a} = \frac{a(c+d+a-b)}{c+a} = \frac{2a(s-b)}{c+a},$$

$$2s' - 2v = a + u - v = a + \frac{a(b-d)}{c+a} = \frac{a(a+b+c-d)}{c+a} = \frac{2a(s-d)}{c+a}.$$

Hence the area Δ' of $\triangle PAB$ is given by

$$\Delta' = \sqrt{s'(s'-a)(s'-u)(s'-v)} = \sqrt{\frac{a(s-a)}{c-a}\,\frac{a(s-c)}{c-a}\,\frac{a(s-b)}{c+a}\,\frac{a(s-d)}{c+a}}. \tag{5}$$

This simplifies to:

$$\Delta' = \frac{a^2}{c^2-a^2}\sqrt{(s-a)(s-b)(s-c)(s-d)}. \tag{6}$$

We have now found the area of $\triangle PAB$.

Step 3: The scale factor in the similarity $\triangle PAB \sim \triangle PDC$ is $k = c/a$. So the area Δ'' of $\triangle PCD$ is $k^2 = c^2/a^2$ times the above expression; that is,

$$\Delta'' = \frac{c^2}{c^2-a^2}\sqrt{(s-a)(s-b)(s-c)(s-d)}. \tag{7}$$

Step 4: The area σ of quadrilateral $ABCD$ is equal to $\Delta'' - \Delta'$. This simplifies to:

$$\sigma = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \tag{8}$$

and we have proved Brahmagupta's formula.

Exercise. Derive the following formulas for u and v:

$$u = \frac{a(ad+bc)}{c^2-a^2}, \qquad v = \frac{a(ab+cd)}{c^2-a^2}.$$

References

[1] http://www-history.mcs.st-andrews.ac.uk/Biographies/Brahmagupta.html
[2] http://en.wikipedia.org/wiki/Brahmagupta
[3] http://www.cut-the-knot.org/Generalization/Brahmagupta.shtml

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