

# Problems for the SENIOR SCHOOL

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**Problem VIII-2-S.1**

Let  $ABCD$  be a parallelogram. Suppose  $K$  is a point such that  $AK = BD$  and let  $M$  be the midpoint of  $CK$ . Prove that  $\angle BMD = 90^\circ$ . [*Tournament of Towns*]

**Problem VIII-2-S.2**

Let  $A$  be a finite non-empty set of consecutive positive integers with at least two elements. Is it possible to partition  $A$  into two disjoint non-empty sets  $X$  and  $Y$  such that the sum of the least common multiples of the numbers in  $X$  and  $Y$  is a power of 2?

**Problem VIII-2-S.3**

The vertices of a prism are coloured using two colours, so that each lateral edge has its vertices differently coloured. Consider all the segments that join vertices of the prism and are not lateral edges. Prove that the number of such segments with endpoints differently coloured is equal to the number of such segments with endpoints of the same colour. [*Romanian Math Competition*]

**Problem VIII-2-S.4**

Let  $a, b, c, d \in [0, 1]$ . Prove that

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+d} + \frac{d}{1+a} + abcd \leq 3.$$

[*Romanian Math Competition*]

**Problem VIII-2-S.5**

Let  $a$  and  $n$  be positive integers such that

$$\text{Frac} \left( \sqrt{n + \sqrt{n}} \right) = \text{Frac} (\sqrt{a}).$$

Prove that  $4a + 1$  is a perfect square. (Here  $\text{Frac}(x) =$  the fractional part of  $x$ .) [*Romanian Math Competition*]

*Keywords: Tournament of towns, Romanian math competition, fractional part, prism, median, perfect square, prime number*

## Solutions of Problems in Issue-VIII-1 (March 2019)

### Solution to problem VIII-1-S.1

*An altitude  $AH$  of triangle  $ABC$  bisects a median  $BM$ . Prove that the medians of the triangle  $ABM$  are side-lengths of a right-angled triangle.*

Let  $AH$  and  $BM$  meet at the point  $K$ , let  $L$  be the midpoint of  $AM$ , and let  $N$  and  $P$  be the projections of  $L$  and  $M$  respectively to  $BC$ . Since  $K$  is the midpoint of  $BM$ , it follows that  $KH$  is a midline of triangle  $BMP$ , i.e.,  $PH = HB$ . On the other hand, by the Thales theorem,  $CP = PH$  and  $PN = NH$ , hence  $N$  is the midpoint of  $BC$ . Therefore  $NK$  is a medial line of triangle  $BMC$ , i.e.  $NK$  is parallel to  $AC$  and  $ALNK$  is a parallelogram. Hence  $LN = AK$ . Also the median from  $M$  in triangle  $AMB$  is a midline of  $ABC$ , hence it is congruent to  $BN$ . Therefore the sides of right-angled triangle  $BNL$  are congruent to the medians of  $ABM$ .

### Solution to problem VIII-1-S.2

*There exists a block of 1000 consecutive positive integers containing no prime numbers, namely,  $1001! + 2, 1001! + 3, \dots, 1001! + 1001$ . Does there exist a block of 1000 consecutive positive integers containing exactly 5 prime numbers?*

Starting with the given block, create a new block by replacing the largest number of the given block with a number that is one less than the smallest number of the given block. In doing so, the number of primes in the block changes by at most one. By the time we reach the first 1000 positive integers the number of primes is a lot more than 5. Thus somewhere along the way we must have had a block of 1000 positive integers with exactly 5 prime numbers.

### Solution to problem VIII-1-S.3

*Initially, the number 1 and two positive numbers  $x$  and  $y$  are written on a blackboard. In each step, we can choose any two numbers on the blackboard, not necessarily different, and write their sum or their difference on the blackboard. We can also choose any non-zero number on the blackboard and write its reciprocal on the blackboard. Is it possible to write on the blackboard, in a finite number of moves, the numbers  $x^2$  and  $xy$ ?*

- (a) We choose the numbers  $x$  and 1 and then write down  $x + 1$  and  $x - 1$ . Then we can write down  $\frac{1}{x-1}$ ,  $\frac{1}{x+1}$  and their difference  $\frac{2}{x^2-1}$ . The reciprocal of this number is  $\frac{x^2-1}{2}$ . Adding this number to itself yields  $x^2 - 1$  and adding 1 to it yields  $x^2$ .
- (b) First write  $x + y$ . By (a), we can write down  $(x + y)^2$ ,  $x^2$  and  $y^2$ . Thereafter, the numbers,  $(x + y)^2 - x^2 = 2xy + y^2$  and  $2xy + y^2 - y^2 = 2xy$ . Finally, write down the reciprocal of  $2xy$  and add it to itself to obtain the reciprocal of  $xy$ . Now  $xy$  can be written by taking the reciprocal of  $\frac{1}{xy}$ .

### Solution to problem VIII-1-S.4

*For which positive integers  $n$  can one find distinct positive integers  $a_1, a_2, \dots, a_n$  such that the number*

$$N = \frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

*is also an integer?*

We shall show that the stated requirement is possible for every positive integer  $n$  except  $n = 2$ .

For  $n = 1$ ,  $N = \frac{a_1}{a_1} = 1$ .

For  $n = 2$ , suppose that  $1 \leq a_1 < a_2$  and assume that  $a_1, a_2$  are coprime. (There is clearly no loss of generality in assuming that the two integers are coprime.) Then if  $N = \frac{a_1}{a_2} + \frac{a_2}{a_1}$  is a positive integer, say  $k$ , then

$$a_1^2 + a_2^2 = ka_1a_2,$$

which shows that  $a_1$  divides  $a_2^2$ . But, since  $a_1$  and  $a_2$  are coprime, this implies  $a_1 = 1$ , which in turn implies  $a_2 = 1$ . A contradiction. Thus, if  $n = 2$ , we cannot find distinct positive integers  $a_1$  and  $a_2$  such that  $N$  is an integer.

For  $n \geq 3$ , let  $a_k = (n-1)^{k-1}$  for  $1 \leq k \leq n$ . These are distinct integers since  $n-1 > 1$  and we have

$$N = \frac{1}{n-1} + \frac{1}{n-1} + \cdots + \frac{1}{n-1} + (n-1)^{n-1} = 1 + (n-1)^{n-1},$$

which is a positive integer. Note that this construction fails if  $n = 1$ , for it yields the same value for all the  $a_i$ .

### Solution to problem VIII-1-S.5

*In triangle  $ABC$ ,  $\angle A = 2\angle B = 4\angle C$ . Their bisectors meet the opposite sides at  $D, E$  and  $F$  respectively. Prove that  $DE = DF$ .*

Let  $I$  be the incentre of  $\triangle ABC$ . Let  $\angle BCI = \theta = \angle ACI$ . Then  $\angle ABI = \angle CBI = 2\theta$  and  $\angle CAI = \angle BAI = 4\theta$ . Hence

$$\angle AIE = \angle BID = \angle BDI = 6\theta, \quad \angle AIF = \angle AFI = 5\theta, \quad \angle AEI = 4\theta.$$

Let  $AI = x$  and  $DI = y$ . Then  $AF = IE = x$  and  $BD = BI = x + y$ .

In  $\triangle BAD$  we have  $AB/AI = DB/DI$ , hence

$$BF = \left( \frac{DB}{DI} - \frac{AF}{AI} \right) \cdot AI = \frac{x^2}{y}.$$

In  $\triangle ABE$  we have  $EA/EI = BA/BI$ , hence

$$AE = (AF + FB) \cdot \frac{EI}{BI} = \frac{x^2}{y} = BF.$$

It follows that  $\triangle EAD$  and  $\triangle FBD$  are congruent to each other, so that  $DE = DF$ .